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Abstract

A simple recipe for generating a complete set of mutually unbiased bases in dimension $2j+1$, with $2j$ integer and $2j+1$ prime, is developed from a single matrix V_a acting on a space of constant angular momentum j and defined in terms of the irreducible characters of the cyclic group C_{2j+1}. This recipe yields an (apparently new) compact formula for the vectors spanning the various mutually unbiased bases. In dimension $(2j+1)^e$, with $2j$ integer, $2j+1$ prime and e positive integer, the use of direct products of matrices of type V_a makes it possible to generate mutually unbiased bases. As two pending results, the matrix V_a is used in the derivation of a polar decomposition of SU(2) and of a FFZ algebra.

PACS: 03.65.Ta; 03.65.Fd; 03.67.-a

Keywords: MUBs; angular momentum; Lie algebra; polar decomposition; deformations; FFZ algebra

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1 Introduction

The notion of mutually unbiased bases (MUBs),[1−23] originally introduced by Schwinger and named by Wootters,[3] is of paramount importance in quantum information theory, especially in quantum cryptography and quantum state tomography. Let us recall that two orthonormal bases $\{|an_\alpha\rangle : n_\alpha = 0, 1, \cdots, d-1\}$ and $\{|bn_\beta\rangle : n_\beta = 0, 1, \cdots, d-1\}$ of a d-dimensional Hilbert space, with an inner product denoted as $\langle\,|\,\rangle$, are said to be mutually unbiased if and only if

$$|\langle an_\alpha|bn_\beta\rangle| = \delta(a,b)\delta(n_\alpha, n_\beta) + [1-\delta(a,b)]\frac{1}{\sqrt{d}}.$$

In dimension d, the maximum number of pairwise MUBs is $d+1$;[1−5] a set consisting of $d+1$ pairwise MUBs is called a complete set. As a matter of fact, the upper bound $d+1$ is attained when d is a prime number or the power of a prime number.[2−9,16] There are numerous ways for constructing complete sets of MUBs,[1−23] most ot them being based on discrete Fourier analysis in Galois fields and Galois rings,[3,9,12,14,16,19,21] discrete Wigner functions,[3,10,21,22] generalized Pauli matrices.[5−8,10] Note also that the existence of MUBs can be related to the problem of finding mutually orthogonal Latin squares[11,15,22] and a solution of the mean King problem.[11,22] Let us also mention that the existence of MUBs has been addressed by various authors from the point of view of finite geometries.[13,15,17,19] Finally, Lie algebra approaches to MUBs have been developed recently.[20,23]

The main aim of this note is to give a simple algorithm for generating MUBs in dimension d where d is a prime number. The case where d is the power of a prime number is briefly examined. The present work constitutes a continuation of the ones in Ref. 23.

2 The Main Results

Let $\epsilon(j)$ be a $(2j+1)$-dimensional Hilbert space of constant angular momentum j (the quantum number j is such that $2j \in \mathbf{N}^*$). An orthonormal basis for $\epsilon(j)$ is provided by the set $\{|j,m\rangle : m = j, j-1, \cdots, -j\}$ where the angular momentum state vectors $|j,m\rangle$, sometimes referred to as spherical or computational or Fock states, are eigenstates of the square J^2 of a generalized angular momentum and its z-component j_z.

Following the suggestion made in Ref. 23 of "redefining the operator U_r" used in a study of SU(2), we introduce the $(2j+1)$-dimensional unitary matrix

$$V_a = \begin{pmatrix} 0 & q^a & 0 & \cdots & 0 \\ 0 & 0 & q^{2a} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & q^{2ja} \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad a \in \{0, 1, \cdots, 2j\},$$

builded on the spherical or standard basis $b_s = (|j,j\rangle, |j,j-1\rangle, \cdots, |j,-j\rangle)$. Here, the parameter q is a rooth of unity defined by

$$q = \exp\left(\mathrm{i}\frac{2\pi}{2j+1}\right).$$

We have the immediate property

$$\mathrm{Tr}\left(V_a^\dagger V_b\right) = (2j+1)\delta(a,b).$$

The matrix V_a is a generalization of the matrix U_r with $r \in \mathbf{R}$ considered in Ref. 24 in the framework of a polar decomposition of SU(2) and used in Ref. 23 for generating MUBs in the cases $d = 2$ and 3. The set $\{V_0, V_1, \cdots, V_{2j}\}$ of the $2j+1$ matrices V_a is constructed from the $2j+1$ irreducible character vectors of the cyclic group C_{2j+1}. Indeed, the nonzero matrix elements of the matrix V_a are given by the irreducible character vector

$$\chi^a = (1, q^a, \cdots, q^{2ja})$$

of C_{2j+1}.

It is straightforward to find the eigenvalues and eigenvectors of V_a. As a result, the spectrum of V_a is non-degenerate. The eigenvector $|jan_\alpha\rangle$ corresponding to the eigenvalue

$$\lambda(jan_\alpha) = q^{ja-n_\alpha}$$

reads

$$|jan_\alpha\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{\frac{1}{2}(j+m)(j-m+1)a+(j+m)n_\alpha} |j,m\rangle, \tag{1}$$

where $n_\alpha = 0, 1, \cdots, 2j$. The $2j+1$ eigenvectors $|jan_\alpha\rangle$ of the matrix V_a generate an orthonormal basis b_a of the space $\epsilon(j)$. For fixed a, the bases b_a and b_s are mutually unbiased. More specifically, we have the following result.

Result 1. In the case where $2j+1$ is a prime integer, the set comprizing the spherical basis b_s and the $2j+1$ bases b_a for $a = 0, 1, \cdots, 2j$ constitute a complete set of $2(j+1)$ MUBs.

At this point, a natural question arises. How to construct a complete set of MUBs for the direct product space $\epsilon(j) \otimes \epsilon(j) \otimes \cdots \otimes \epsilon(j)$ (with e factors) of dimension $d = (2j+1)^e$, where $2j+1$ is prime and e is an integer greater or equal to 2? The answer follows from the following result.

Result 2. In the case where $2j+1$ is a prime integer, the eigenvectors of the matrices

$$W_{a_1 a_2 \cdots a_e} = V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_e}, \quad a_i \in \{0, 1, \cdots, 2j\}, \quad i = 1, 2, \cdots, e,$$

together with the d-dimensional computational basis can be arranged to form a complete set of $d + 1 = (2j+1)^e + 1$ MUBs.

The proofs of Results 1 and 2 can be obtained from an adaptation of the proofs in Refs. 5-7, 12 and 21. The term “arranged” in Result 2 means that auxilliary matrices need to be introduced in order to deal with the degeneracy problem.

As a corollary of Result 1, we otain the sum rule

$$\left| \sum_{k=0}^{d-1} q^{\frac{1}{2}k(d-k)(a-b)+k(n_\alpha - n_\beta)} \right| = d\delta(a,b)\delta(n_\alpha, n_\beta) + \sqrt{d}[1 - \delta(a,b)],$$

with

$$q = \exp\left(\mathrm{i}\frac{2\pi}{d}\right), \quad a, b \in \{0, 1, \cdots, d-1\}, \quad n_\alpha, n_\beta \in \{0, 1, \cdots, d-1\},$$

where d is a prime number.

3 Two Related Results

We would like to outline two Lie-like aspects of our approach.

First, we can find a polar decomposition of the shift operators j_+ and j_- of the Lie group SU(2) in terms of the unitary operator v_a associated to the matrix V_a. The operator v_a satisfies

$$v_a|j,m\rangle = q^{(j-m)a}[1-\delta(m,j)]|j,m+1\rangle + \delta(m,j)|j,-j\rangle$$

for $m = j, j-1, \cdots, -j$. Following Refs. 23 and 24, let us define the Hermitean operator h through

$$h|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m\rangle.$$

We can show that the linear operators

$$j_+ = hv_a, \quad j_- = v_a^\dagger h, \quad j_z = \frac{1}{2}(h^2 - v_a^\dagger h^2 v_a)$$

have the following action

$$j_\pm|j,m\rangle = q^{\pm(j\mp m+\frac{1}{2}\mp\frac{1}{2})a}\sqrt{(j-m)(j+m+1)}|j,m\pm1\rangle, \quad j_z|j,m\rangle = m|j,m\rangle \qquad (2)$$

on the standard state vector $|j,m\rangle$ for $m = j, j-1, \cdots, -j$. As a consequence, we get

$$[j_z, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = 2j_z.$$

Hence, the operators j_+, j_- and j_z span the Lie algebra of SU(2). This result is to be compared with similar results obtained in Refs. 21 and 23-25 without the occurrence of the parameter a. It is to be emphasized that this result holds for any value of a ($a = 0, 1, \cdots, 2j$). However, note that the action of $j_\pm$ on $|j,m\rangle$ depends on a. The Condon and Shortley phase convention used in atomic spectroscopy amounts to take $a = 0$ in Eq. (2).

Second, the cyclic character of the irreducible representations of C_{2j+1} renders possible to express V_a in function of V_0. In fact, we have

$$V_a = V_0 Z^a,$$

where

$$Z = \mathrm{diag}(1, q, \cdots, q^{2j}).$$

The matrices V_a and Z have an interesting property, namely, they q-commute in the sense that

$$V_a Z - qZV_a = 0.$$

By defining

$$T_m = q^{\frac{1}{2}m_1 m_2} V_a^{m_1} Z^{m_2}, \quad m = (m_1, m_2) \in \mathbf{N}^{*2},$$

we easily obtain the commutator

$$[T_m, T_n] = 2\mathrm{i} \sin\left(\frac{\pi}{2j+1} m \wedge n\right) T_{m+n},$$

where

$$m \wedge n = m_1 n_2 - m_2 n_1, \quad m + n = (m_1 + n_1, m_2 + n_2),$$

so that the linear operators T_m span the FFZ infinite dimensional Lie algebra introduced by Fairlie, Fletcher and Zachos.[26] The latter result parallels the ones obtained, on one hand, from a study of k-fermions and of the Dirac quantum phase operator through a q-deformation of the harmonic oscillator[27] and, on the other hand, from an investigation of correlation measure for finite quantum systems.[25]

4 Closing Remarks

In the case where $d = 2j + 1$ is a prime number, Result 1 provides us with a simple mean for generating a complete set of $d + 1$ MUBs from the knowledge of a single matrix, viz., the matrix V_a. It should be noted that when $2j + 1$ is not a prime number, Eq. (1) can be used for spanning MUBs as well; however, in that case, it is not possible to generate a complete set of MUBs.

The main interest of our approach relies on the fact that MUBs can be constructed from a simple generic matrix V_a and yields calculations easily codable on a computer. In addition, the matrix V_a turns out to be of physical interest and plays an important role in the polar decomposition of SU(2) and for the derivation of the FFZ algebra.

These matters, inherited from a q-deformation approach to symmetry and supersymmetry,[27,28] will be developed in a forthcoming paper in a larger context involving MUBs, useful in quantum information, and symmetry adapted bases, useful in molecular physics and quantum chemistry.

Acknowledgements

One of the author (M.R.K.) is grateful to Prof. D. Ellinas for useful correspondence.

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